The isoperimetric inequality: from antiquity to Steiner

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Introduction

Problem: In $\mathbb{R}^n$, what is the largest region that can be enclosed by a boundary of fixed size?

Introduction

Simplest problem: In $\mathbb{R}^2$, what region of fixed perimeter L maximizes area A?

Zenodorus' polygon proof

Theorem: Isosceles triangles maximize area when compared to non-isosceles triangles

Heron's formula:

$$A = \frac{\sqrt{L(L-2a)(L-2b)(L-2c)}}{4},$$

where $L = a + b + c$.

Zenodorus' polygon proof

Fix c and L. Then by the Arithmetic Mean - Geometric Mean inequality,

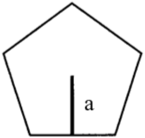
$$\begin{aligned} c &= \frac{(L-2a)+(L-2b)}{2} \\ &\geq \sqrt{(L-2a)(L-2b)} \end{aligned}$$

thus

$$c\sqrt{L(L-2c)} \geq 4A$$

i.e., A is bounded by $c\sqrt{L(L-2c)}/4$, and maximized if $L-2a = L-2b$, that is, when $a = b$.

Zenodorus' polygon proof



$$A_{n\text{-gon}} := \frac{aL}{2}$$

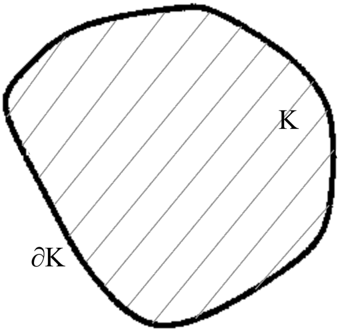
where L = perimeter, a = apothem.

Idea: Increasing the number of sides of a regular n-gon increases its area.

Isoperimetric inequality

In $\mathbb{R}^2 : \dfrac{A}{L^2} \leq \dfrac{1}{4\pi}$, with equality only for a circle.

Calculus of variations



$$\gamma(t) = (x(t), y(t))$$

Calculus of variations

$$L(t,\gamma,\dot{\gamma}) := \int_a^b ||\dot{\gamma}(t)||\,\mathrm{d}t,$$

$$I(t,\gamma,\dot{\gamma}) := \frac{y(t)\dot{x}(t) - x(t)\dot{y}(t)}{2},$$

$$A(t,\gamma,\dot{\gamma}) := \int_{\partial K} I(t,\gamma,\dot{\gamma})\,\mathrm{d}t = \frac{1}{2}\int_a^b y(t)\dot{x}(t) - x(t)\dot{y}(t)\,\mathrm{d}t$$

(by Green's theorem)

Calculus of variations

Euler-Lagrange:

$$\frac{\partial I}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial I}{\partial \dot{q}} = 0$$

Idea: Use Lagrange Multiplier method and Euler–Lagrange to maximize A under fixed L.

Calculus of variations

$$\frac{1}{2}\int_a^b y(t)\dot{x}(t) - x(t)\dot{y}(t)\,\mathrm{d}t + \lambda\int_a^b \sqrt{\dot{x}(t)^2+\dot{y}(t)^2}\,\mathrm{d}t$$

$$\Downarrow$$

$$\begin{cases} \frac{\dot{y}(t)}{2} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y(t)}{2} + \frac{\lambda\dot{x}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}}\right) = 0 \\ \frac{\dot{x}(t)}{2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{x(t)}{2} + \frac{\lambda\dot{y}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}}\right) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{y}(t) + \lambda\ddot{x}(t) = 0 \\ \dot{x}(t) - \lambda\ddot{y}(t) = 0 \end{cases} \quad\Rightarrow\quad \begin{cases} x(t) = x_0 + \lambda\cos t \\ y(t) = y_0 + \lambda\sin t \end{cases}$$

Steiner symmetrization

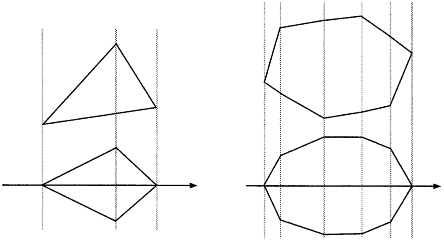
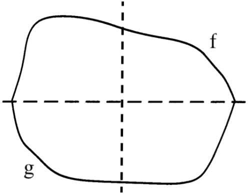


Figure: Adjusting each slice to create a symmetric figure

Steiner symmetrization

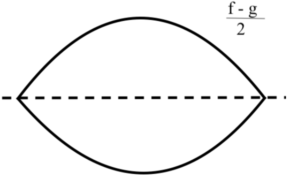


$$A := \int_a^b f(t)\,\mathrm{d}t - \int_a^b g(t)\,\mathrm{d}t$$

$$L := \int_a^b \sqrt{1 + f'(t)^2}\,\mathrm{d}t + \int_a^b \sqrt{1 + g'(t)^2}\,\mathrm{d}t$$

This is a strategy that can be used in any dimension.

Steiner symmetrization



$$A_{sym} = 2\int_a^b \frac{f(t) - g(t)}{2}\,\mathrm{d}t$$

$$L_{sym} = 2\int_a^b \sqrt{\frac{(f'(t) - g'(t))^2}{4} + 1}\,\mathrm{d}t$$

Steiner symmetrization

$A_{sym} = A$ and $L_{sym} \leq L$,

$$\frac{A}{L^2} \leq \frac{A_{sym}}{L_{sym}^2},$$

and repeated Steiner symmetrizations on a region converge to a ball.

Conclusion: A/L^2 is maximized for a ball.