On de Rham Cohomology

Alejandro Escoto
Mentor: Vlassis Mastrantonis

University of Maryland, College Park

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Overview

1. Chain complexes and cohomology.
2. Manifolds and differential forms.
3. de Rham cohomology of a manifold and examples
4. The Mayer–Vietoris sequence and examples
5. Smooth homotopy and homotopy axiom for de Rham cohomology

Chain complexes

To begin, we define a cochain complex.

Definition

A *cochain complex* C is a collection of vector spaces $\{C^k\}_{k \in \mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^{k+1}$ such that for all k, $d_k \circ d_{k-1} = 0$.

$$\cdots \longrightarrow C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} C^{k+1} \longrightarrow \cdots$$

Figure: A cochain complex

Cohomology

The condition $d_k \circ d_{k-1} = 0$ is equivalent to $\ker d_k \supset \operatorname{im} d_{k-1}$. Therefore,

$$H^k(C) := {\ker d_k} \Big/ {\operatorname{im} d_{k-1}},$$

is the *k-th cohomology* of C. In particular,

$$\dim H^k(C) = \dim \ker d_k - \dim \operatorname{im} d_{k-1}.$$

Exactness

A chain complex C is *exact* when

$$H^k(C) = 0,$$

for all k. That is, $\ker d_k = \mathrm{im} d_{k-1}$.

Zig–Zag Lemma

Given a short exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{k+1} & \xrightarrow{i_{k+1}} & B^{k+1} & \xrightarrow{j_{k+1}} & C^{k+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_k} & & \uparrow{\scriptstyle d_k} & & \uparrow{\scriptstyle d_k} & & \\
0 & \longrightarrow & A^{k} & \xrightarrow{i_{k}} & B^{k} & \xrightarrow{j_{k}} & C^{k} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_{k-1}} & & \uparrow{\scriptstyle d_{k-1}} & & \uparrow{\scriptstyle d_{k-1}} & & \\
0 & \longrightarrow & A^{k-1} & \xrightarrow{i_{k-1}} & B^{k-1} & \xrightarrow{j_{k-1}} & C^{k-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

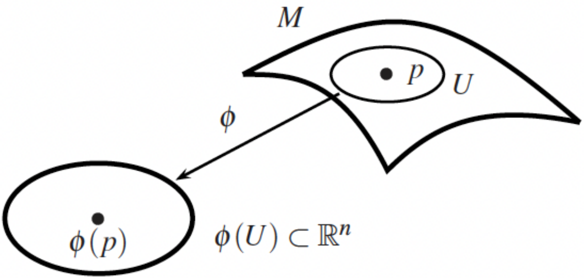
Zig–Zag lemma

This induces a long exact sequence in cohomology,

$$\cdots \xrightarrow{j^*} H^{k-1}(\mathcal{C}) \xrightarrow{d^*} H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C}) \xrightarrow{d^*} H^{k+1}(\mathcal{A}) \xrightarrow{i^*} \cdots,$$

Manifolds

Manifolds can be thought of as generalizations of curves and surfaces that are locally euclidean.



Differential forms

A differential form is given by

$$\omega = \sum a_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where $a_{i_1 \dots i_k}$ are functions on a manifold, where $(x^1, \dots, x^n)$ are local coordinates.
We denote the space of C^∞ differential k-forms on a manifold M by $\Omega^k(M)$

Exterior derivative

The exterior derivative of ω is locally given by

$$\mathrm{d}\omega = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$$

Is such that $\mathrm{d} \circ \mathrm{d} = 0$

In particular, the exterior derivative takes k-forms to k+1-forms

de Rham cohomology

The collection of C^∞ differential forms on M, $\Omega^*(M)$ with the exterior derivative gives rise to the following cochain complex

$$\cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \longrightarrow \cdots$$

The k-th de Rham cohomology vector space of M is given by

$$H^k_{\mathrm{dR}}(M) := H^k(\Omega^*(M)).$$

Easy examples of de Rham cohomology

$$H^k_{\mathrm{dR}}(\mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

This is because the cochain complex of differential forms on $\mathbb{R}$ is

$$\Omega^0(\mathbb{R}) \simeq \mathbb{R} \xrightarrow{d_0} \Omega^1(\mathbb{R}) \simeq \mathbb{R} \xrightarrow{d_1} 0 \longrightarrow \cdots$$

and by the fundamental theorem of calculus, for any 1-form $\omega = f(x)\,\mathrm{d}x$ and $F(x) := \int_0^x f(t)\,\mathrm{d}t$, $\mathrm{d}F = f(x)\,\mathrm{d}x = \omega$. Therefore,

$$H^0_{\mathrm{dR}}(\mathbb{R}) := \ker d_0 = \mathbb{R}, \quad H^1_{\mathrm{dR}}(\mathbb{R}) := {\ker d_1} \big/ {\mathrm{im}\, d_0} = 0.$$

Mayer-Vietoris sequence

These maps and the sets of differential forms on $M, U, V, U \cap V$ give rise to the following short exact sequence of cochain complexes

$$0 \longrightarrow \Omega^*(M) \xrightarrow{\ i\ } \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\ j\ } \Omega^*(U \cap V) \longrightarrow 0$$

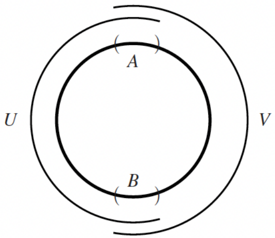
$i(\omega) = (\omega|_U, \omega|_V)$, and $j(\alpha, \beta) = \alpha|_{U\cap V} - \beta|_{U\cap V}$.

Mayer-Vietoris sequence

$$\cdots \xrightarrow{j^*} H^{k-1}(U \cap V) \xrightarrow{d^*} H^k(M) \xrightarrow{i^*} H^k(U) \oplus H^k(V) \xrightarrow{j^*} H^k(U \cap V) \xrightarrow{d^*} H^{k+1}(M) \xrightarrow{i^*} \cdots .$$

Some examples: S^1

The following open cover of S^1,



by the Mayer–Vietoris, gives rise to the following exact sequence,

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ i\ } \mathbb{R} \oplus \mathbb{R} \xrightarrow{\ j\ } \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1_{\mathrm{dR}}(S^1) \longrightarrow 0$$

Some examples: S^n

From which we can compute

$$H^k_{\mathrm{dR}}(S^1) = \begin{cases} \mathbb{R}, & k = 0, 1 \\ 0, & k > 1. \end{cases}$$

By induction, it follows from the Mayer–Vietoris

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R}, & \text{k=0, n,} \\ 0, & \text{othwerwise.} \end{cases}$$

Homotopy

Let M,N be manifolds. Two C^∞ *maps* $f, g : M \to N$ are smoothly homotopic if there is a map

$$F : M \times \mathbb{R} \to N$$

such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = g(x)$$

In this case we write

$$f \sim g$$

Homotopy

Definition

A map $f : M \to N$ is a *homotopy equivalence* if it has a *homotopy inverse* $g : N \to M$ such that

$$f \circ g \sim 1_M \text{ and } g \circ f \sim 1_N$$

If there is a *homotopy equivalence* of two Manifolds, we say they have the same *homotopy type* with the notion of homotopy equivalence we introduce the Homotopy Axiom for de Rham cohomolgy it will be stated as a theorem.

Theorem

Two manifolds that are homotopy equivalent have the same de Rham cohomology.

This is powerful tool we can use to compute the de Rham cohomology of a manifold

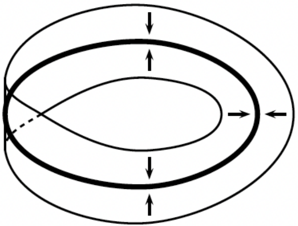
Homotopy axiom example: $\mathbb{R}^n$

All the Euclidean spaces $\mathbb{R}^n$ deformation retract to a point, therefore

$$H^k_{\mathrm{dR}}(\mathbb{R}^n) = H^k_{\mathrm{dR}}(\text{point}) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Homotopy axiom example: Möbius band

The Möbius band M is a 2-dimensional non-orientable manifold that is deformation retracts to a circle S^1.



Therefore,

$$H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}, & k = 0, 1, \\ 0, & k > 1. \end{cases}$$