The 2-dimensional Mahler conjecture

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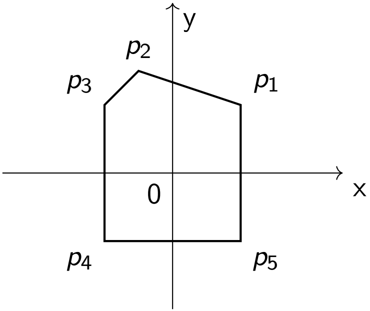
Overview

1. The polar of a convex polytope.
2. A lower bound for triangles.
3. The sliding argument.

Convex Polytopes

Let $P = \mathrm{conv}\{p_1, \ldots, p_n\}$ be a convex polytope on the plane with vertices $p_1, \ldots, p_n$. Here, $\mathrm{conv}\{p_1, \ldots, p_n\}$ denotes the smallest convex set containing $p_1, \ldots, p_n$.



Convex Polytopes

For a convex polytope P containing the origin in its interior $0 \in \operatorname{int} P$, one can describe P as the intersection of half-planes

$$P = \bigcap_{i=1}^{n} \{(x, y) \in \mathbb{R}^2 : a_i x + b_i y \leq 1\}.$$

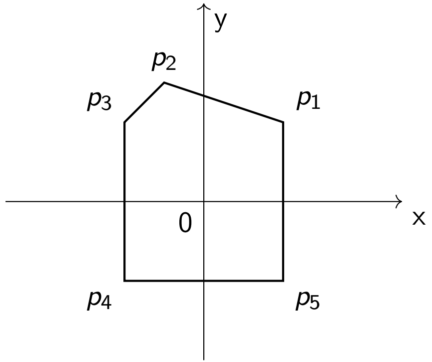
This generates n new points $Q_i = (a_i, b_i)$ that lie in the direction normal to the edge $[p_i, p_{i+1}]$.

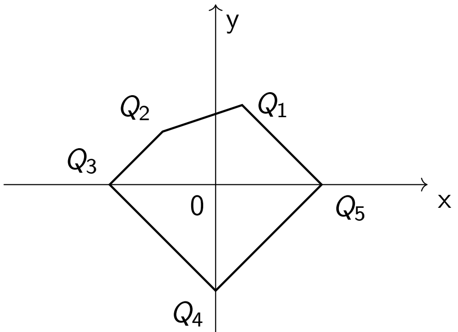
Using this one can form a new polytope called the *polar polytope* defined by

$$P^\circ := \mathrm{conv}\{Q_1, \ldots, Q_n\},$$

or equivalently

$$P^\circ = \{(x, y) \in \mathbb{R}^2 : ax + by \leq 1, \text{ for all } (a, b) \in P\}.$$





Convex Polytopes

Polarity is a duality operation: the polar of the polar polytope is the original body, i.e., $(P^\circ)^\circ = P$.

Moreover, by definition, the bigger P is the smaller P° becomes and vice-versa. As a result, an interesting question occurs: Is

$$\mathcal{M}(P) := |P||P^\circ|,$$

the product of their volumes, bounded?

Goal

Our goal in this talk is to prove the following theorem, following Mahler's argument.

Theorem

For all convex polytopes $P \subset \mathbb{R}^2$,

$$\mathcal{M}(P) \geq \frac{27}{4}.$$

Triangles

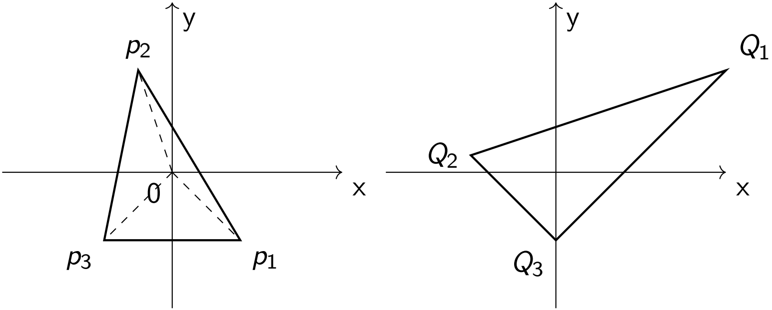
The first step is to prove the bound for triangles. Let $\Delta = \mathrm{conv}\{p_1, p_2, p_3\}$ be a triangle with vertices $p_i = (x_i, y_i)$, and

$$\Lambda_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \Lambda_2 = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}, \Lambda_3 = \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

the volume of the parallelogram with vertices $0, p_1, p_2, p_1 + p_2$. Then,

$$|\Delta| = \frac{1}{2}(\Lambda_1 + \Lambda_2 + \Lambda_3).$$

Triangles



Triangles

Moreover, Δ is bounded by the lines

$$(y_2 - y_1)x - (x_2 - x_1)y = \Lambda_1,$$
$$(y_3 - y_2)x - (x_3 - x_2)y = \Lambda_2,$$
$$(y_1 - y_3)x - (x_1 - x_3)y = \Lambda_3.$$

As a result, the polar triangle

$$\Delta^\circ = \mathrm{conv}\{Q_1, Q_2, Q_3\}$$

where

$$Q_i = \left(\frac{y_{i+1} - y_i}{\Lambda_i}, \frac{x_i - x_{i+1}}{\Lambda_i}\right),$$

Triangles

As before, for

$$\tilde{\Lambda}_i := \begin{vmatrix} Q_i \\ Q_{i+1} \end{vmatrix} = \frac{1}{\Lambda_i\Lambda_{i+1}} \begin{vmatrix} y_{i+1} - y_i & x_i - x_{i+1} \\ y_{i+2} - y_{i+1} & x_{i+1} - x_{i+2} \end{vmatrix} = \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{\Lambda_i\Lambda_i},$$

the volume

$$|\Delta^\circ| = \frac{1}{2}(\tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \tilde{\Lambda}_3) = \frac{(\Lambda_1 + \Lambda_2 + \Lambda_3)^2}{2\Lambda_1\Lambda_2\Lambda_3}.$$

Lemma

For $\Delta \subset \mathbb{R}^2$ a triangle, $\mathcal{M}(\Delta) \geq 27/4$.

Proof. If $0 \notin \operatorname{int} \Delta$, the polar $|\Delta^\circ| = +\infty$. Therefore, we may assume $0 \in \operatorname{int} \Delta$. By the previous formulas,

$$|\Delta| = \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2},$$
$$|\Delta^\circ| = \frac{(\Lambda_1 + \Lambda_2 + \Lambda_3)^2}{2\Lambda_1\Lambda_2\Lambda_3}.$$

Thefore, by the AM–GM inequality,

$$\mathcal{M}(\Delta) := |\Delta||\Delta^\circ| = \frac{(\Lambda_1 + \Lambda_2 + \Lambda_3)^3}{4\Lambda_1\Lambda_2\Lambda_3} \geq \frac{3^3\Lambda_1\Lambda_2\Lambda_3}{4\Lambda_1\Lambda_2\Lambda_3} = \frac{27}{4}.$$

□

The sliding

It remains to show the following:

Theorem

For $\Pi_m \subset \mathbb{R}^2$ a polytope with $m \geq 4$ vertices, there exists a polytope Π_{m-1} with $m-1$ vertices such that

$$\mathcal{M}(\Pi_{m-1}) < \mathcal{M}(\Pi_m).$$

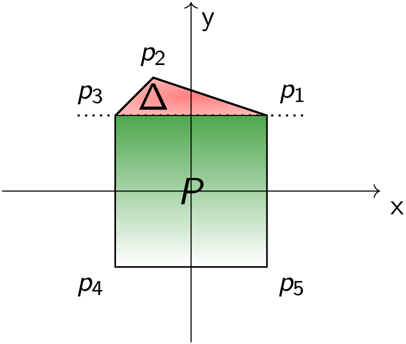
This would imply the desired bound, since by the previous Theorem, the Mahler volume is minimized among triangles, for which we know that $\mathcal{M} \geq 27/4$ as desired.

The sliding

Proof of Theorem. For technical reasons we restrict to $m \geq 5$ and $0 \in \Pi_m$. Split

$$\Pi_m = P \cup \Delta$$

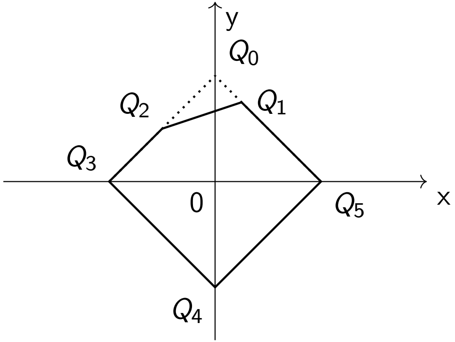
to a disjoint union as below,



Note that since $P \subset \Pi_m$ the polars $\Pi_m^\circ \subset P^\circ$. In particular,

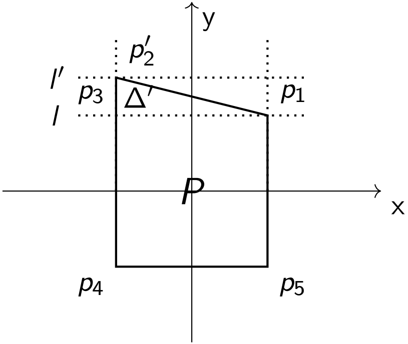
$$\tilde{\Delta} := P^\circ \setminus \Pi_m^\circ,$$

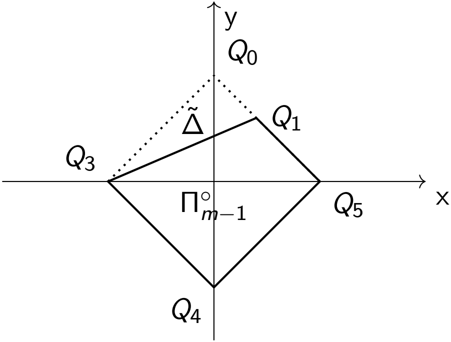
is also a triangle because the points Q_0, Q_1, Q_5 are colinear, and Q_0, Q_2, Q_3 are also colinear.



Mahler's idea was to slide p_2 along the line parallel to $[p_1, p_3]$ so that p_3 becomes colinear to p_2 and p_5, or p_1 becomes colinear to p_2 and p_m, such that

$$|\Delta| = |\Delta'|.$$





The sliding

Note that

$$\Pi_{m-1} := P \cup \Delta'$$

is a polytope with $m-1$ vertices so that

$$|\Pi_m| = |\Pi_{m-1}|,$$

so it remains to show that

$$|\Pi_{m-1}^\circ| < |\Pi_m^\circ|.$$

However,

$$|\Pi_{m-1}^\circ| = |P^\circ| - |\tilde{\Delta}|.$$

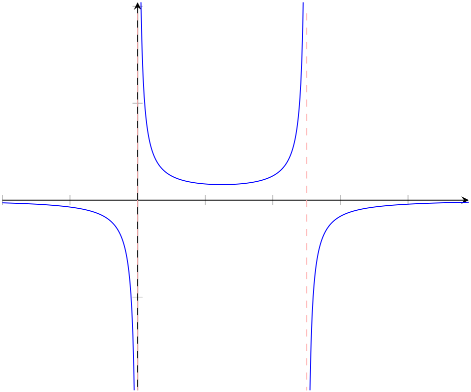
The sliding

$$|\tilde{\Delta}| = \frac{1}{2}\left(\begin{vmatrix} Q_1 \\ Q_0 \end{vmatrix} + \begin{vmatrix} Q_0 \\ Q_2 \end{vmatrix} + \begin{vmatrix} Q_2 \\ Q_1 \end{vmatrix}\right) = \frac{(\Lambda_1 + \Lambda_2 - \Lambda_0)^2}{2\Lambda_0\Lambda_1\Lambda_2}.$$

$$= \frac{2(|\Delta| - \Lambda_0)^2}{\Lambda_0\Lambda_1(2|\Delta| - \Lambda_1 - \Lambda_0)} = \frac{c}{\Lambda_1(b - \Lambda_1)},$$

The sliding

As we slide p_2 along the line parallel to $[p_1, p_3]$ one may compute volume of $\tilde{\Delta}$ to change as shown in the graph below:



The sliding

Note that this has unique minimum and increases from there. Therefore, if we move at the right direction we can make sure that either p_1 or p_3 will "disappear" while the volume of $\widetilde{\Delta}$ will have increased, i.e., the volume of the polar has decreased. □

The End

Thank you!